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Generation and some nonclassical properties of a nonlinear finite-dimensional pair coherent state

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Abstract
We propose a scheme for generating a nonlinear finite-dimensional pair coherent state of the vibrational motion of an ion in a two-dimensional trap. It is a type of the correlated two-mode states but in finite dimensions. Based on the resonant ion–cavity interaction, we propose a scheme to generate these states revealing their connection with the converter type of interaction and investigate some of their nonclassical properties.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is at the heart of the current development of quantum information processing [1]. Entanglement-assisted communication can enlarge the channel capacity [2] and enhance channel efficiency [3]. Entanglement may play a key role in communication security [4]. In quantum computation, of course, qubits are massively entangled. The generation and characterization of entanglement have been studied extensively. Coherent states and their variants and generalizations have been extensively studied over the last four decades (a comprehensive review of this development can be found in [5]). Subsequently, the notion was generalized in various ways. Motivations to generalize the concept have arisen from symmetry considerations [6], dynamics [7] and algebraic aspects [8]. A generalized class of the conventional coherent state called the nonlinear coherent states or the f-coherent state [9] has been constructed.

On the other hand, pair coherent states (PCS) are regarded as an important type of correlated two-mode states, which possess prominent nonclassical properties. Such states denoted by $|\zeta, q\rangle$ are eigenstates of the operator pair $(\hat{a}\hat{b})$ and the number difference $(\hat{n}_a - \hat{n}_b)$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators of the field modes and $\hat{n}_a = \hat{a}^\dagger\hat{a}$ and $\hat{n}_b = \hat{b}^\dagger\hat{b}$. These states satisfy

$$\hat{a}\hat{b}|\zeta,q\rangle = \zeta|\zeta,q\rangle \qquad \text{and} \qquad (\hat{n}_a - \hat{n}_b)|\zeta,q\rangle = q|\zeta,q\rangle. \tag{1}$$

The experimental realization of such nonclassical states is of practical importance. Agarwal [10] suggested that the optical PCS can be generated via the competition of four-wave mixing and two-photon absorption in a nonlinear medium. Another scheme has been suggested for generating vibrational pair coherent states via the motion of a trapped ion in a two-dimensional trap [11].

An ion confined in an electromagnetic trap can be regarded as a particle with quantized centre-of-mass motion in a harmonic potential. Exciting or de-exciting the internal atomic states of the trapped ion by a classical laser driving field changes the external states of the ion motion, as atomic-stimulated absorption and emission processes are always accompanied by momentum exchange of the laser field with the ion. If the vibrational amplitude of the ion is much smaller than the laser wavelength, i.e., in the Lamb–Dicke limit [12], and the driving field is tuned to one of the vibrational sidebands of the atomic transition, then this model can be simplified to a form similar to the Jaynes–Cummings model (JCM) [13] in which the quantized radiation field is replaced by the quantized centre-of-mass motion of the ion. As the coupling between the vibrational modes and the external environment is extremely weak, dissipative effects which are inevitable from cavity damping in the optical regime can be significantly suppressed for the ion motion. This unique feature thus makes it possible to realize cavity QED experiments without using an optical cavity. Following this approach, nonclassical vibrational states of the trapped ions such as Fock [14], squeezed [15] and Schrödinger cat states [16] have been proposed and observed [17].

On the other hand, the finite-dimensional PCS has been studied recently by Obada and Khalil [18] as the eigenstate of the pair operators $\left(\hat{a}^\dagger\hat{b} + \frac{\zeta^{q+1}(\hat{a}\hat{b}^\dagger)^q}{(q!)^2}\right)$ and the sum of the photon number operators for the two modes ($\hat{Q} = \hat{n}_a + \hat{n}_b$). In the present paper we develop this idea and introduce a nonlinear finite-dimensional state (NPCS) as the eigenstate of the pair operators $\left(f_1(\hat{n}_a)\hat{a}^\dagger f_2(\hat{n}_b)\hat{b} + \frac{\zeta^{q+1}\left(\hat{a}\frac{1}{f_1(\hat{n}_a)}\frac{1}{f_2(\hat{n}_b)}\hat{b}^\dagger\right)^q}{(q!)^2}\right)$ and $(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})$ for the two modes, namely,

$$\left(f_1(\hat{n}_a)\hat{a}^\dagger\hat{b}f_2(\hat{n}_b) + \frac{\zeta^{q+1}\left(\hat{a}\frac{1}{f_1(\hat{n}_a)}\frac{1}{f_2(\hat{n}_b)}\hat{b}^\dagger\right)^q}{(q!)^2}\right)|\zeta,q\rangle = \zeta|\zeta,q\rangle, \qquad \hat{Q}|\zeta,q\rangle = q|\zeta,q\rangle, \tag{2}$$

where the parameter ζ is a complex variable while the parameter q is an integer. The expansion of this state in the two-mode states $|n_a,n_b\rangle = |n_a\rangle\otimes|n_b\rangle$, where $|n_s\rangle$ is the Fock state for the mode s ($s = a$ or b), takes the form

$$|\zeta,q\rangle = N_q\sum_{n=0}^{q}\zeta^n\sqrt{\frac{(q-n)!}{q!n!}}\frac{f_1(q-n)!}{f_1(q)!f_2(n)!}|q-n,n\rangle, \tag{3}$$

where $f(n)! = f(0)\cdot f(1)\cdots f(n)$ and $f(0) = 1$, the normalization constant N_q is given by

$$N_q = \left[\sum_{n=0}^{q}|\zeta|^{2n}\frac{(q-n)!}{q!n!}\left(\frac{f_1(q-n)!}{f_1(q)!f_2(n)!}\right)^2\right]^{\frac{-1}{2}}. \tag{4}$$

Because of the appearance of the operators $\hat{a}^\dagger\hat{b}$ or $\hat{a}\hat{b}^\dagger$ in this form and the functions $f_1(\hat{n}_a)$ and $f_2(\hat{n}_b)$, it may be legitimate to call it a finite-dimensional nonlinear pair coherent state or converter state.

Once we have introduced this class of nonlinear finite-dimensional pair coherent states, we wish to discuss some of their statistical properties. The results that we are going to present stem from a new approach to the above state. Subsequently, we shall examine the sub-Poissonian distribution and the phase properties of the state (3). Therefore, the generation scheme for the mentioned state is demonstrated in the next section.

2. Generation scheme

In this section, we concern ourselves with the context of trapped ions. Since ions can be trapped very efficiently and their entanglement with the environment is extremely weak, trapped ions have advantages for many purposes such as preparing various types of nonclassical states (see, e.g., [17–24]), simulating nonlinear interactions [25], demonstrating quantum phase transitions [26, 27], establishing quantum search algorithms [28] and so on. The most promising merit of trapped ion systems is perhaps the possibility of implementing scalable quantum computers [29] in which a number of ions are involved [30–32]. Nevertheless, many tasks can still be done even with a single ion. For instance, a controlled-NOT quantum logic gate can be performed just by a single trapped ion [33–36]. Here, we propose an experimental scheme to generate the state of equation (3) in the vibrionic motion of an ion which is trapped in real two-dimensional (2D) space.

The specification of the operators $\left(f_1(\hat{n}_a)\hat{a}^\dagger f_2(\hat{n}_b)\hat{b} + \frac{\zeta^{q+1}\left(\hat{a}\frac{1}{f_1(\hat{n}_a)}\frac{1}{f_2(\hat{n}_b)}\hat{b}^\dagger\right)^q}{(q!)^2}\right)$ is subject to the generation schemes within the framework of the motion of a trapped ion in a two-dimensional harmonic potential. Consider a single ion trapped in a 2D harmonic potential with frequencies ν_1 (in the x-direction), ν_2 (in the y-direction) in interaction with three laser fields propagating in the same direction tuned respectively to the electronic transition ω_0 of the ion and to the vibrational sideband of frequency taken as follows: the first vibrational sideband has the frequency $(\nu_2 - \nu_1)$ lower than that transition, but the second vibrational sideband has the frequency $q(\nu_1 - \nu_2)$ higher than that transition. The Hamiltonian of this system is written as

$$\begin{aligned} H = {} & \nu_1\hat{a}^\dagger\hat{a} + \nu_2\hat{b}^\dagger\hat{b} + \frac{\omega_0}{2}\sigma_z + \underline{\mu}[\{\underline{E}_0 \exp \mathrm{i}(k_1x + k_2y - \omega_0 t + \phi_0) \\ & + \underline{E}_1 \exp \mathrm{i}(k_1x + k_2y - [\omega_0 - (\nu_2 - \nu_1)t] + \phi_1) \\ & + \underline{E}_2 \exp \mathrm{i}(k_1x + k_2y - [\omega_0 - q(\nu_1 - \nu_2)]t + \phi_2)\}\sigma_+ + \mathrm{h.c.}]. \end{aligned} \tag{5}$$

We denote by $\hat{a}$ and $\hat{b}$ the annihilation operators of the quantized bosons that describe the vibrational motion in the two dimensions x and y. The operators $\sigma_+(\sigma_-)$ and σ_z are the raising (lowering) and the population inversion operators of the electronic states of the two-level ion. $\underline{\mu}$ is the dipole matrix element and k_s $(s = 1, 2)$ are the components of the wave vectors of the driving laser fields of amplitudes E_0, E_1 and E_2 with phases ϕ_0, ϕ_1 and ϕ_2, respectively. The quantized centre-of-mass position $\hat{x}$ and $\hat{y}$ can be written as

$$\hat{x} = \Delta x(\hat{a} + \hat{a}^\dagger), \qquad \hat{y} = \Delta y(\hat{b} + \hat{b}^\dagger), \tag{6}$$

with Δx and Δy are the standard deviation for $\hat{x}$ and $\hat{y}$ in the ground state of the harmonic potential. We may use a vibrational rotating wave approximation and neglect the terms with fast oscillations [11, 37]. Thus, the interactions Hamiltonian is simplified to

$$\begin{aligned} H_{\mathrm{int}} = {} & \exp\left[-\frac{(\eta_1^2 + \eta_2^2)}{2}\right]\left[\sigma_+\left\{\Omega_0 \exp(\mathrm{i}\phi_0) \sum_{m_1,m_2} \frac{(\mathrm{i}\eta_1)^{2m_1}(\mathrm{i}\eta_2)^{2m_2}}{(m_1!)^2(m_2!)^2}\hat{a}^{\dagger m_1}\hat{a}^{m_1}\hat{b}^{\dagger m_2}\hat{b}^{m_2}\right.\right. \\ & + \Omega_1 \exp(\mathrm{i}\phi_1) \sum_{m_1,m_2} \frac{(\mathrm{i}\eta_1)^{2m_1+1}(\mathrm{i}\eta_2)^{2m_2+1}}{m_1!(m_1+1)!m_2!(m_2+1)!}\hat{a}^{\dagger m_1+1}\hat{a}^{m_1}\hat{b}^{\dagger m_2}\hat{b}^{m_2+1} \end{aligned}$$

$$+\,\Omega_2\exp(\mathrm{i}\phi_2)\sum_{m_1,m_2}\frac{(\mathrm{i}\eta_1)^{2m_1+q}(\mathrm{i}\eta_2)^{2m_2+q}}{m_1!(m_1+q)!m_2!(m_2+q)!}\hat{a}^{\dagger m_1}\hat{a}^{m_1+q}\hat{b}^{\dagger m_2+q}\hat{b}^{m_2}\Bigg\}+\mathrm{h.c.}\Bigg]. \tag{7}$$

$|\Omega_0| = |\underline{\mu}\cdot\underline{E}_0|$, $|\Omega_1| = |\underline{\mu}\cdot\underline{E}_1|$ and $|\Omega_2| = |\underline{\mu}\cdot\underline{E}_2|$ are the Rabi frequencies related to the different laser fields and η_s are the Lamb–Dicke parameters, where $\eta_1 = k_1\Delta x$, $\eta_2 = k_2\Delta y$ [11]. It should be noted that $\hat{n}_1+\hat{n}_2$ is a constant of motion for the Hamiltonian (7). The terms in the parenthesis in (7) can be summed in terms of the associated Laguerre polynomials; thus equation (7) is given as follows:

$$H_{\mathrm{int}} = \lambda(f_1(\hat{n}_a)\hat{a}^\dagger f_2(\hat{n}_b)\hat{b} + \zeta^{q+1}\hat{a}^q f_3(\hat{n}_a)\hat{b}^{\dagger q} f_4(\hat{n}_b) - \zeta)\sigma_+ + \mathrm{h.c.} \tag{8}$$

where

$$\begin{aligned} f_1(\hat{n}_a) &= \frac{L^1_{\hat{n}_a-1}(\eta_1^2)}{(\hat{n}_a)L^0_{\hat{n}_a}(\eta_1^2)}, & f_2(\hat{n}_b) &= \frac{L^1_{\hat{n}_b}(\eta_2^2)}{(\hat{n}_b+1)L^0_{\hat{n}_b}(\eta_2^2)}, \\ f_3(\hat{n}_1) &= \frac{(\hat{n}_a-q)!L^1_{\hat{n}_a-q}(\eta_1^2)}{(\hat{n}_a)!L^0_{\hat{n}_a}(\eta_1^2)}, & f_4(\hat{n}_b) &= \frac{\hat{n}_b!L^1_{\hat{n}_b}(\eta_2^2)}{(\hat{n}_b+q)!L^0_{\hat{n}_b+q}(\eta_2^2)}, \end{aligned} \tag{9}$$

$$\begin{aligned} \lambda &= -\Omega_1\eta_1\eta_2 L^0_{\hat{n}_a}(\eta_1^2)L^0_{\hat{n}_b}(\eta_2^2)\exp\left[-\frac{(\eta_1^2+\eta_2^2)}{2}+\mathrm{i}\phi_1\right], \\ \zeta &= \frac{-\Omega_0\exp i(\phi_0-\phi_1)}{\Omega_1\eta_1\eta_2}, \end{aligned} \tag{10}$$

and $L^m_n(x)$ are associated Laguerre polynomials given by $L^m_n(x) = \sum_{r=0}^n \binom{n+m}{n-r}\frac{(-1)^r}{r!}x^r$. While Ω_2 is related to the other parameters through the formula $\Omega_2 = \frac{\zeta^{q+1}\Omega_1}{(-1)^{q-1}(\eta_1\eta_2)^{q-1}f_1(q)!f_2(q)!}$, therefore the parameters ζ and q are controlled by the amplitudes and phases of the applied laser fields and the Lamb–Dicke parameters. In the experiments performed on $^9Be^+$ ion with laser beam containing ≈ 1 nw of power at 313 nm, the Lamb–Dicke parameter η is calculated to be ≈ 0.23. Thus, using this estimate for η_1 and η_2 put $\eta_1\eta_2 \approx 0.05$. For the values $|\zeta| \approx \eta_1\eta_2$ and for arbitrary q, then $\Omega_0 \sim \Omega_1(\eta_1\eta_2)^2$, $\Omega_2 \sim \Omega_1\frac{(\eta_1\eta_2)^2}{f_1(q)!f_2(q)!}$ which gives $\Omega_0 \sim \frac{\Omega_1}{400}$, $\Omega_2 \sim \frac{\Omega_1}{400 f_1(q)!f_2(q)!}$. Thus, the value for E_1 has to be two orders of magnitude higher than E_0 and E_2. Since $\Omega_i = \underline{\mu}\cdot\underline{E}_i = \mu\, E_i\cos(\theta_i)$ $(i = 0, 1, 2)$ the angle θ_i can be used to reduce the estimate for E_i. This means that moderate values for E_0 and E_2 and strong value of E_1 are sufficient to produce such state with arbitrary q for $|\zeta| \approx \eta_1\eta_2$. However, for larger values of $|\zeta|$ the number q must attain large values for appropriate laser fields.

For generating the state of equation (3) let us look at the master equation for the density matrix under spontaneous emission with energy dissipation rate γ which is given by [11]

$$\frac{\partial\bar{\rho}}{\partial t} = -\mathrm{i}[H_{\mathrm{int}},\rho] + \frac{\gamma}{2}[2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-]. \tag{11}$$

The stationary solution $\bar{\rho}_s$ for this master equation is obtained by setting $\frac{\partial\bar{\rho}}{\partial t} = 0$. A solution $\bar{\rho}_s$ can be given as

$$\bar{\rho}_s = |g\rangle|\zeta\rangle\langle\zeta|\langle g|, \tag{12}$$

with $|g\rangle$ being the electronic ground state $(\sigma_-|g\rangle = 0, \langle g|\sigma_+ = 0)$ and $|\zeta\rangle$ is the vibration eigenstate that satisfies $H_{\mathrm{int}}|\zeta\rangle = 0$. It is straightforward to show that $|\zeta\rangle$ belongs to the class of states considered in (2). To tailor the Hamiltonian of any nonlinear multi-quanta JCM a scheme of using a number of lasers has been presented to produce such interaction [38, 39]. It is to be mentioned that the nonlinear JCM has been realized experimentally [40].

3. Relations to other states

3.1. Relation to $SU(2)$ group

The operators J_x, J_y and J_z are defined as

$$
\begin{aligned}
J_x &= \frac{\left(f_1(\hat{n}_a)\hat{a}^\dagger f_2(\hat{n}_b)\hat{b} + \hat{a}\frac{1}{f_1(\hat{n}_a)}\hat{b}^\dagger\frac{1}{f_2(\hat{n}_b)}\right)}{2},\\
J_y &= \frac{\left(f_1(\hat{n}_a)\hat{a}^\dagger f_2(\hat{n}_b)\hat{b} - \hat{a}\frac{1}{f_1(\hat{n}_a)}\hat{b}^\dagger\frac{1}{f_2(\hat{n}_b)}\right)}{2i},\\
J_z &= \frac{(\hat{n}_a - \hat{n}_b)}{2},
\end{aligned}
\tag{13}
$$

which satisfy the commutation relations $[J_x, J_y] = \mathrm{i}J_z$, $[J_y, J_z] = \mathrm{i}J_x$ and $[J_z, J_x] = \mathrm{i}J_y$. It is useful to introduce the following operators:

$$
J_+ = J_x + \mathrm{i}J_y = f_1(\hat{n}_a)\hat{a}^\dagger f_2(\hat{n}_b)\hat{b}, \qquad J_- = J_x - \mathrm{i}J_y = \hat{a}\frac{1}{f_1(\hat{n}_a)}\hat{b}^\dagger\frac{1}{f_2(\hat{n}_b)}. \tag{14}
$$

Then, we have the commutation relation

$$
[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_z. \tag{15}
$$

Furthermore, the operator

$$
\hat{C}_2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = \left(\frac{\hat{n}_a + \hat{n}_b}{2}\right)\left(\frac{\hat{n}_a + \hat{n}_b}{2} + 1\right). \tag{16}
$$

These operators can be thought of as operation under Lie algebra with the generators J_i. It is to be noted that J_x and J_y are not Hermitian operators and hence J_+ is not the Hermitian conjugate of J_-. The operator $\hat{C}_2$ commutes with all the generators of the Lie algebra and in the language of group theory is known as a Casimir operator. The state (3) is a eigenstate for the operator $\hat{C}_2$ with eigenvalue $\frac{q}{2}\left(\frac{q}{2} + 1\right)$. The unitary irreducible representations of the $SU(2)$ are just the familiar angular momentum states $|j, m\rangle$ satisfying the relations

$$
\begin{aligned}
&\hat{C}_2|j, m\rangle = j(j+1)|j, m\rangle, \qquad J_z|j, m\rangle = m|j, m\rangle\\
&J_+|j, m\rangle = |f_1(j+m+1)||f_2(j-m-1)|\sqrt{(j+m+1)(j-m)}|j, m+1\rangle,\\
&J_-|j, m\rangle = \frac{\sqrt{(j+m)(j-m+1)}}{|f_1(j+m)||f_2(j-m)|}|j, m+1\rangle,\\
&\qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \quad m = -j, -j+1, \ldots, j.
\end{aligned}
\tag{17}
$$

Note that the representations are finite dimensional, the dimension for a given j being $2j + 1$. Now if we take $q = 2j$, the state (3) takes the following form:

$$
\begin{aligned}
|\zeta, 2j\rangle &= N_{2j}\sum_{n=0}^{2j}\zeta^n\frac{f_1(2j-n)!}{f_1(2j)!f_2(n)!}\sqrt{\frac{(2j-n)!}{2j!n!}}|2j-n, n\rangle\\
&= N_{2j}\sum_{n=-j}^{j}\zeta^{n+j}\frac{f_1(j-n)!}{f_1(2j)!f_2(n+j)!}\sqrt{\frac{(j-n)!}{2j!(n+j)!}}|j-n, n+j\rangle,
\end{aligned}
\tag{18}
$$

which is eigenstate of the operator $\hat{C}_2$ with eigenvalue $j(j+1)$.

3.2. Exponential form

The state $|\zeta, q\rangle$ of equation (3) may be cast as

$$|\zeta, q\rangle = N_q \sum_{n=0}^{q} \zeta^n \frac{(q-n)!\hat{a}^n \hat{b}^{\dagger n}}{q!n!} \frac{f_1(q-n)!}{f_1(q)!f_2(n)!}|q, 0\rangle. \tag{19}$$

One can show that

$$[\hat{a}\hat{b}^\dagger g(\hat{n}_a, \hat{n}_b)]^n = \hat{a}^n \hat{b}^{\dagger n} \prod_{m=0}^{n-1} g(\hat{n}_a - m, \hat{n}_b + m). \tag{20}$$

Here, $g(\hat{n}_a, \hat{n}_b)$ is an arbitrary function of $\hat{n}_a$ and $\hat{n}_b$. Then using equation (19), with

$$g(\hat{n}_a, \hat{n}_b) = \frac{\zeta}{f_1(\hat{n}_a) f_2(\hat{n}_b)\hat{n}_a}, \tag{21}$$

the state $|\zeta, q\rangle$ is finally written in the exponential form

$$|\zeta, q\rangle = N_q \sum_{n=0}^{\infty} \frac{\left[\hat{a}\hat{b}^\dagger \frac{\zeta}{f_1(\hat{n}_a) f_2(\hat{n}_b)\hat{n}_a}\right]^n}{n!} |q, 0\rangle = N_q \exp\left[\hat{a}\hat{b}^\dagger \frac{\zeta}{f_1(\hat{n}_a) f_2(\hat{n}_b)\hat{n}_a}\right]|q, 0\rangle. \tag{22}$$

3.3. Bell states

Entanglement is an essential resource for many applications in quantum information science such as quantum superdense coding [41, 42], quantum teleportation [43–48], quantum cryptography [49–51] and quantum computing [52, 53], most of these applications are based on the maximally entangled two-particle quantum states called Bell states. The maximally entangled single-phonon number states are defined as $|\psi\rangle = |1, 0\rangle$ or $|0, 1\rangle$, two-phonon number states as $|\psi\rangle = |1, 1\rangle$ and null-photon states as $|\psi\rangle = |0, 0\rangle$. We can generate the maximally entangled states by taking the nonlinear functions $f_1(\hat{n}_a) = \hat{I}$ and $f_2(\hat{n}_b) = \hat{I}$ and the parameter q takes the values 1, 2, 0, respectively.

$$\begin{aligned} |1, 1\rangle &= \frac{1}{\zeta N_2\sqrt{2}}[|\zeta, 2\rangle - |-\zeta, 2\rangle] \qquad & |0, 1\rangle &= \frac{1}{2\zeta N_1}[|\zeta, 1\rangle - |-\zeta, 1\rangle] \\ |1, 0\rangle &= \frac{1}{2N_1}[|\zeta, 1\rangle + |-\zeta, 1\rangle] \qquad & |0, 0\rangle &= |0, 0\rangle \end{aligned} \tag{23}$$

where N_q is given by equation (4), the maximal entangled states are defined as follows:

$$\psi_\pm = \frac{1}{\sqrt{2}}(|1, 1\rangle \pm |0, 0\rangle), \qquad \varphi_\pm = \frac{1}{\sqrt{2}}(|1, 0\rangle \pm |0, 1\rangle). \tag{24}$$

Thus, the maximally entangled states $\psi_\pm, \varphi_\pm$ which play an important role in quantum measurement theory can be constructed from the states $|\zeta, q\rangle$.

4. Nonclassical effects

The experimental feasibility of models involving more than one mode in a high-Q cavity has been more or less considered by many authors [54]. It is worthwhile remarking that investigating such models goes beyond an intrinsic theoretical interest because new generation of high-Q electromagnetic cavities, covering a wide wavelength range, is realizable today [54, 55]. Thus, in the following subsections we investigate the influence of the controlling parameters q, η_1 and η_2 on the nonclassical behaviour of the modes where, in particular, the sub-Poissonian distribution and the phase distribution are discussed.

4.1. Sub-Poissonian distribution

In this section, we shall evaluate the correlation function to discuss the phenomenon of sub-Poissonian distribution. This phenomenon can be measured by photon detectors based on photoelectric effect. It is well known that sub-Poissonian statistics is characterized by the fact that the variance of the photon number $\langle(\Delta\hat{n}_i)^2\rangle$ is less than the average photon number $\langle\hat{a}_i^\dagger\hat{a}_i\rangle = \langle\hat{n}_i\rangle$. This can be expressed by means of the normalized second-order correlation function for the mode z in a quantum state $|\zeta, q\rangle$ [56] as follows:

$$g_z^{(2)}(\zeta) = \frac{\langle\zeta, q|\hat{n}_z(\hat{n}_z - 1)|\zeta, q\rangle}{\langle\zeta, q|\hat{n}_z|\zeta, q\rangle^2}, \qquad \forall z = a, b \tag{25}$$

where

$$\begin{aligned}\langle\zeta, q|\hat{n}_a(\hat{n}_a - 1)|\zeta, q\rangle &= N_q^2\sum_{n=0}^{q}|\zeta|^{2n}\left(\frac{f_1(q-n)!}{f_1(q)!f_2(n)!}\right)^2\frac{(q-n)!}{q!n!}(q-n)(q-n-1),\\ \langle\zeta, q|\hat{n}_b(\hat{n}_b - 1)|\zeta, q\rangle &= N_q^2\sum_{n=0}^{q}|\zeta|^{2n}\left(\frac{f_1(q-n)!}{f_1(q)!f_2(n)!}\right)^2\frac{(q-n)!}{q!n!}n(n-1),\end{aligned} \tag{26}$$

and

$$\begin{aligned}\langle\zeta, q|\hat{n}_a|\zeta, q\rangle &= N_q^2\sum_{n=0}^{q}|\zeta|^{2n}\left(\frac{f_1(q-n)!}{f_1(q)!f_2(n)!}\right)^2\frac{(q-n)!}{n!}(q-n),\\ \langle\zeta, q|\hat{n}_b|\zeta, q\rangle &= N_q^2\sum_{n=0}^{q}|\zeta|^{2n}\left(\frac{f_1(q-n)!}{f_1(q)!f_2(n)!}\right)^2\frac{(q-n)!}{n!}n,\end{aligned} \tag{27}$$

where $f_1(q-n)$ and $f_2(n)$ are given by equation (9). The function $g_z^{(2)}(\zeta)$ given by (23) for the mode z serves as a measure of the deviation from the Poissonian distribution that corresponds to coherent states with $g_z^{(2)}(\zeta) = 1$. If $g_z^{(2)}(\zeta) < 1$ (>1) the distribution is called sub (super)-Poissonian, if $g_z^{(2)}(\zeta) = 2$, the distribution is called thermal and when $g_z^{(2)}(\zeta) > 2$ it is called super-thermal.

To reveal the physical content of the state, we plot $g_a^{(2)}(\zeta)$ against $|\zeta|$. In the first case when we take $\eta = 0$, we show that when $q = 0$ or 1 the function $g_a^{(2)}(\zeta) = 0$ due to the fact that the states present are either vacuum or one photon and for both of them $g^{(2)}(\zeta)$ is zero. For the effectiveness we take $q = 3$, it is to be observed that the state starts at $g_a^{(2)}(0) = \frac{2}{3}$ and for a short interval of $|\zeta|$ the function $g_a^{(2)}(\zeta)$ has full sub-Poissonian distribution. Also super-Poissonian behaviour appears for higher values of ζ and its behaviour is almost like the thermal distribution as observed in figure 1(a). In figure 1(a) we take $q = 4, 5$, we find that the function starts at $\frac{3}{4}$ and $\frac{4}{5}$, respectively, as has been studied earlier in [18]. This is because it appears that we have the Fock state $|q\rangle$ present in this case when $\zeta \to 0$ and $g_a^{(2)}(\zeta) = \frac{q-1}{q}$. In this basis, we see that $g_a^{(2)}(\zeta) < 1$ for a short range of ζ. When the parameter ζ is increased further, the state $|\zeta, q\rangle$ exhibits super-Poissonian behaviour and for large values of $|\zeta|$ the state reaches super-thermal state behaviour because for $\zeta \to \infty$ we get the limit $g_b^{(2)}(\zeta) = \frac{4(q-1)}{q}$. The nonclassical nature of the state is apparent, when one takes the value $q = 2$ where the function $g_a^{(2)}(\zeta) < 1$ as shown in figure 1(a). On the other hand, when we take $q > 2$ the function $g_a^{(2)}(\zeta) > 2$ for higher values of ζ.

As soon as one takes the nonlinear functions $f_1(q-n)$ and $f_2(n)$ into consideration and adjusts the parameters $\eta = \eta_1 = \eta_1 = 0.3$ in figure 1(b), one can see that the starting points are unchanged for the three curves, but the interval of $|\zeta|$ for the full sub-Poissonian and super-Poissonian distributions increases. The super-thermal state behaviour is also appearing,

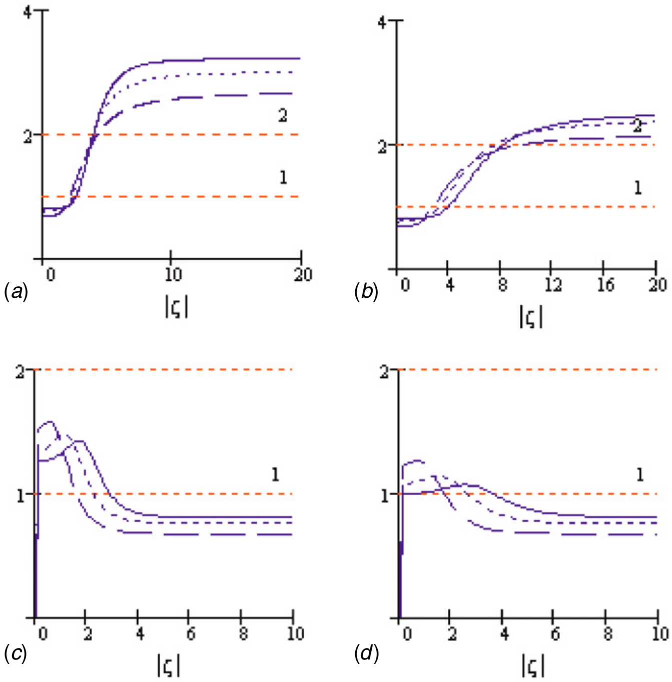

Figure 1. The sub-Poissonian function as a function of $|\zeta|$: (*a*) for mode a and $\eta = 0$, (*b*) for mode b and $\eta = 0$, (*c*) for mode a and $\eta = 0.3$, (*d*) for mode b and $\eta = 0.3$, where the solid curve is for $q = 3$, the dotted curve is for $q = 4$, the dashed curve is for $q = 5$.

but the maximum values for the curves are decreasing comparing with the above case as observed in figure 1(*b*).

Further we consider the function $g_b^{(2)}(\zeta)$ for the second mode. For the parameters $\eta = 0$, q takes the values 3, 4 and 5 and we find that the function $g_b^{(2)}(\zeta)$ starts at 1.5, 0.75 and 1.25, respectively. Because the condition between the two modes $(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})$ is constant, thus when we take the limits as $\zeta \to 0$ we get the limit $g_b^{(2)}(\zeta) = \frac{q}{q-1}$. We see that $g_b^{(2)}(\zeta)$ has a decreasing trend and so for sufficiently large values of $|\zeta|$ it shows sub-Poissonian behaviour because for $\zeta \to \infty$ we get the limit $g_b^{(2)}(\zeta) = \frac{q-1}{q}$. For further increase of q, the state $|\zeta, q\rangle$ exhibits full sub-Poissonian behaviour (see figure 1(*c*)). We note that the super-Poissonian distribution interval increases by increasing the parameter q. As it is exhibited by figures 1(*a*) and (*c*), the modes a and b behave differently for small values of $|\zeta|$ and also for large values of $|\zeta|$. However, both modes may show sub-Poissonian behaviour. For example when we take $|\zeta| = \sqrt{2}$ and $q = 2$, it is found that $g_a^{(2)}(\zeta) = g_b^{(2)}(\zeta) = \frac{2}{3}$ which means sub-Poissonian behaviour in both modes, see [18]. When we take the nonlinearity parameter $\eta = 0.3$ into account, we find that when q is small there exists a short interval of $|\zeta|$ where the function $g_b^{(2)}(\zeta)$ reaches super-Poissonian state behaviour; the distribution is lowered gradually to sub-Poissonian behaviour as observed in figure 1(*d*).

4.2. Phase properties

The quantum properties of the radiation field can be investigated under different points of view. Therefore, we continue our progress and devote the present section to considering and discussing the phase distribution for the states (3). For this reason, it is convenient to

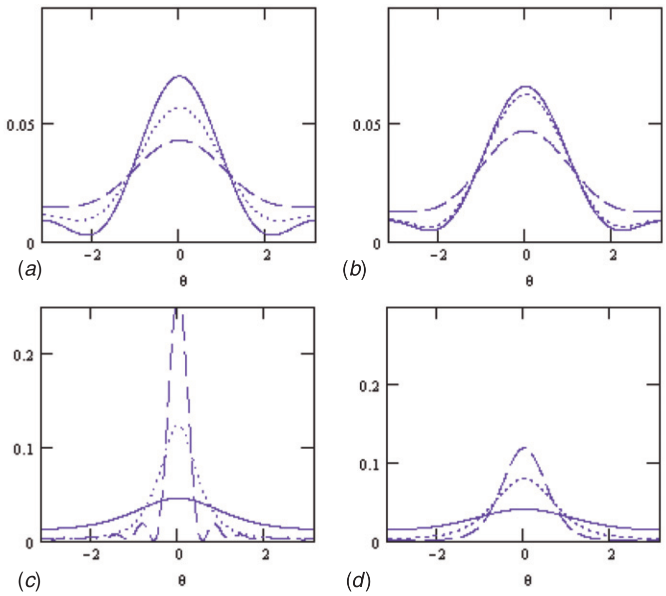

Figure 2. The phase distribution $P_{\zeta,q}(\theta)$ against the angle $\theta = (\theta_2 - \theta_1)$: (*a*) $q = 1$ and $\eta = 0$, (*b*) $q = 10$ and $\eta = 0$, (*c*) $q = 1$ and $\eta = 0.3$, (*d*) $q = 10$ and $\eta = 0.3$, where the solid curve is for $\zeta = 1$, the dotted curve is for $\zeta = 3$ and the dashed curve is for $\zeta = 5$.

use the phase formalism introduced by Barnett and Pegg [56–59]. It is well known that the phase operator is defined as the projection operator on a particular phase state multiplied by the corresponding value of the phase. Therefore, one can find that the Pegg–Barnett phase distribution function $P_{\zeta,q}(\theta_1, \theta_2)$ is given by [46]

$$P_{\zeta,q}(\theta_1,\theta_2) = \frac{|N_q|^2}{(2\pi)^2}\sum_{n,m}\frac{\zeta^n f_1(q-n)!}{f_1(q)!f_2(n)!}\frac{\zeta^{*m} f_1(q-m)!}{(f_1(q)!f_2(m)!)^2}\sqrt{\frac{(q-n)!(q-m)!}{q!n!q!m!}} \times \exp[\mathrm{i}[(q-n)-(q-m)]\theta_1 + \mathrm{i}(n-m)\theta_2]. \tag{28}$$

Therefore, the phases distribution function can be written as

$$P_{\zeta,q}(\theta_1,\theta_2) = \frac{|N_q|^2}{(2\pi)^2}\left|\sum_n \frac{\zeta^n f_1(q-n)!}{f_1(q)!f_2(n)!}\sqrt{\frac{(q-n)!}{q!n!}}\exp[\mathrm{i}n\theta]\right|^2, \qquad -\pi \leqslant \theta \leqslant \pi, \qquad \theta = \theta_2 - \theta_1 \tag{29}$$

which is normalized according to $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P(\theta_1,\theta_2,\zeta)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 = 1$. Due to the correlation between the two modes, the phase distribution depends on the difference between the phases modes. In the figures, we plot $P_{\zeta,q}(\theta)$ against the angle $\theta = \theta_2 - \theta_1$ for different values of the parameters q, $|\zeta|$ and η.

To discuss the phase behaviour for the present state, we have plotted the function against the phase angle θ for $\eta = 0$ and different values of q. Generally, for very small (large) values of $|\zeta|$ the state (3) almost represents a Fock state and hence the information about the phase is lost. As $|\zeta|$ increases partial coherent phase states result and the phase distribution shows a peak. This peak is concerted at $\theta = 0$ and the distribution is symmetric around this peak. For $q = 2$, plotted in figure 2(*a*), it is observed that $P_{\zeta,q}(\theta)$ starts at $P_{\zeta,q}(-\pi) = 0, 0.01, 0.016$

when $|\zeta| = 1, 3$ and 5, respectively. The maxima for the distribution at $\theta = 0$ decrease by increasing $|\zeta|$. In figure 2(a), we take into account the parameter η as equal 0.3, we see that the maximum values of the previous curves are decreased as observed in figure 2(b).

In figure 2(c), we take a large value for the parameters ($\eta = 0$, $q = 10$) and the same values of $|\zeta| = (1, 3, 5)$. We see that the function $P_{\zeta,q}(\theta)$ starts at $P(-\pi) = 0.013, 0.002, 0.003$ when $|\zeta| = 1, 3$ and 5, respectively. The maxima for the distribution at $\theta = 0$ decrease by increasing the value of $|\zeta|$. However, this increase turns to a decrease for larger values of $|\zeta|$. The maximum value for $P_{\zeta,q}(0)$ shifts to higher values of $|\zeta|$ as q increases. As soon as one takes the nonlinear functions $f_1(q - n)$ and $f_2(n)$ into consideration and adjusts the parameters $\eta = 0.3$, we observe that a peak around $\theta = 0$ develops and increases by decreasing the parameter $|\zeta|$ as observed in figure 2(d). The behaviour of the nonlinearity functions for the phase distribution in this case is the same as in the first case (small values of q).

5. Conclusion

In this work, we have studied an extension of a nonlinear finite-dimensional pair coherent state and proposed a scheme for generating a correlated two-mode finite-dimensional state in the vibrational motion of a trapped ion in two-dimensional harmonic potential. These states generated by this scheme are stable because they appear in a steady regime in which the ion has fully relaxed to its ground state. If the vibrational state of motion of the ion is initially formed in this state, then the steady state of the system is a pure state given by a product of the atomic ground state with a state (3) of the vibrational motion. In this case, the three parameters, ζ, q and η, that characterize the two-mode nonlinear finite-dimensional states are determined by the intensities and phases of the driving lasers, the Lamb–Dicke parameter and by the sum of the phonon number of the two vibrational modes. Based on recent techniques the present scheme could be realized experimentally [40]. The effect of the nonlinearity function is shown for the sub-Poissonian and phase distributions. The behaviour of the sub-Poissonian distribution function depends on the values of nonlinearity function and q parameter. Comparisons between the nonlinear finite-dimensional pair coherent state and the standard finite-dimensional pair coherent state have been made for the different phenomena. These states may find applications in the fields of quantum optics and quantum information.

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Appendix

In this appendix we give the derivation of equations (17). We note that the relation between the pair (j, m) and the pair $(\hat{n}_a, \hat{n}_b)$ is

$$2j = \hat{n}_a + \hat{n}_b, \qquad 2m = \hat{n}_a - \hat{n}_b. \tag{A.1}$$

The matrix elements of J_+ can be derived as follows:

$$J_+|j, m\rangle = k_{j,m}|j, m + 1\rangle. \tag{A.2}$$

By using the Hermitian conjugate we find that

$$\langle j, m|J_+^\dagger = \bar{k}_{j,m}\langle j, m + 1| \tag{A.3}$$

then

$$\langle j, m|J_+^\dagger J_+|j, m\rangle = |k_{j,m}|^2. \tag{A.4}$$

From equation (14) and its Hermitian conjugate we show that

$$\begin{aligned}\langle j, m|J_+^\dagger J_+|j, m\rangle &= |f_1(n_a)|^2|f_2(n_b)|^2(n_a + 1)n_b \\ &= |f_1(j + m + 1)|^2|f_2(j - m - 1)|^2(j + m + 1)(j - m),\end{aligned} \tag{A.5}$$

by comparing (A.4) with (A.5) we find that

$$k_{j,m} = |f_1(j + m + 1)||f_2(j - m - 1)|\sqrt{(j + m + 1)(j - m)}. \tag{A.6}$$

In a similar way we can prove that

$$J_-|j, m\rangle = \frac{\sqrt{(j + m)(j - m + 1)}}{|f_1(j + m)||f_2(j - m)|}|j, m + 1\rangle. \tag{A.7}$$

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